ド•モアブルの定理クイズ

$\boxed{1}$ (1) $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^4=\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

(2) $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^{-6}=\cos(-\pi)+i\sin(-\pi)=-1$

解説

$\boxed{2}$ $(\sqrt{3}+i)^{12}$ を計算せよ。

解答 4096

解説

$\sqrt{3}+i=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ であるから

$$(\sqrt{3}+i)^{12}=2^{12}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^{12}$$
$$=2^{12}(\cos 2\pi+i\sin 2\pi)=2^{12}\cdot 1=4096$$

$\boxed{3}$ $(\sqrt{3}-i)^{-3}$ を計算せよ。

解答 $\frac{1}{8}i$

解説

$\sqrt{3}-i=2\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}$ であるから

$$(\sqrt{3}-i)^{-3}=2^{-3}\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}^{-3}$$
$$=2^{-3}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=\frac{1}{8}i$$

$\boxed{4}$ 次の式を計算せよ。

(1) $(1-\sqrt{3}i)^6$　(2) $(-1+i)^5$　(3) $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^{-9}$

解答 (1) 64　(2) $4-4i$　(3) i

解説

(1) $1-\sqrt{3}i=2\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}$ であるから

$$(1-\sqrt{3}i)^6=2^6\{\cos(-2\pi)+i\sin(-2\pi)\}$$
$$=2^6\cdot 1=64$$

(2) $-1+i=\sqrt{2}\left(\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi\right)$ であるから

$$(-1+i)^5=(\sqrt{2})^5\left(\cos\frac{15}{4}\pi+i\sin\frac{15}{4}\pi\right)$$
$$=4\sqrt{2}\left(\cos\frac{7}{4}\pi+i\sin\frac{7}{4}\pi\right)$$
$$=4\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=4-4i$$

(3) $\frac{\sqrt{3}}{2}+\frac{i}{2}=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$ であるから

$$\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^{-9}=\cos\left(-\frac{3}{2}\pi\right)+i\sin\left(-\frac{3}{2}\pi\right)=i$$

$\boxed{5}$ 次の式を計算せよ。

(1) $(1+i)^7$　(2) $\left(\frac{1-\sqrt{3}i}{2}\right)^8$　(3) $\left(\frac{2}{1-\sqrt{3}i}\right)^6$　(4) $\left(\frac{-1+i}{\sqrt{3}+i}\right)^{12}$

解答 (1) $8-8i$　(2) $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$　(3) 1　(4) $-\frac{1}{64}$

解説

(1) $1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ であるから

$$(1+i)^7=(\sqrt{2})^7\left(\cos\frac{7}{4}\pi+i\sin\frac{7}{4}\pi\right)$$
$$=8\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)$$
$$=8-8i$$

(2) $\frac{1-\sqrt{3}i}{2}=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)$ であるから

$$\left(\frac{1-\sqrt{3}i}{2}\right)^8=\cos\left(-\frac{8}{3}\pi\right)+i\sin\left(-\frac{8}{3}\pi\right)$$
$$=\cos\left(-\frac{2}{3}\pi\right)+i\sin\left(-\frac{2}{3}\pi\right)$$
$$=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

(3) $1-\sqrt{3}i=2\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}$ であるから

$$\frac{2}{1-\sqrt{3}i}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

よって $\left(\frac{2}{1-\sqrt{3}i}\right)^6=\cos 2\pi+i\sin 2\pi=1$

(4) $-1+i=\sqrt{2}\left(\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi\right)$, $\sqrt{3}+i=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ であるから

$$\frac{-1+i}{\sqrt{3}+i}=\frac{\sqrt{2}\left(\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi\right)}{2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}$$

よって $\left(\frac{-1+i}{\sqrt{3}+i}\right)^{12}=\left(\frac{\sqrt{2}}{2}\right)^{12}\cdot\frac{\cos 9\pi+i\sin 9\pi}{\cos 2\pi+i\sin 2\pi}$

$$=-\frac{1}{64}$$

$\boxed{6}$ 次の式を計算せよ。[各 10 点]

(1) $\left\{2\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)\right\}^6$　(2) $\left(\frac{3-\sqrt{3}i}{2}\right)^8$

解答 (1) $\left\{2\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)\right\}^6=2^6\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=32+32\sqrt{3}i$

(2) $\frac{3-\sqrt{3}i}{2}=\sqrt{3}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=\sqrt{3}\left(\cos\frac{11}{6}\pi+i\sin\frac{11}{6}\pi\right)$ であるから

$$\left(\frac{3-\sqrt{3}i}{2}\right)^8=\left\{\sqrt{3}\left(\cos\frac{11}{6}\pi+i\sin\frac{11}{6}\pi\right)\right\}^8=(\sqrt{3})^8\left(\cos\frac{44}{3}\pi+i\sin\frac{44}{3}\pi\right)$$
$$=-\frac{81}{2}+\frac{81\sqrt{3}}{2}i$$

解説

(1) $\left\{2\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)\right\}^6=2^6\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=32+32\sqrt{3}i$

(2) $\frac{3-\sqrt{3}i}{2}=\sqrt{3}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=\sqrt{3}\left(\cos\frac{11}{6}\pi+i\sin\frac{11}{6}\pi\right)$ であるから

$$\left(\frac{3-\sqrt{3}i}{2}\right)^8=\left\{\sqrt{3}\left(\cos\frac{11}{6}\pi+i\sin\frac{11}{6}\pi\right)\right\}^8=(\sqrt{3})^8\left(\cos\frac{44}{3}\pi+i\sin\frac{44}{3}\pi\right)$$
$$=-\frac{81}{2}+\frac{81\sqrt{3}}{2}i$$

$\boxed{7}$ $\left(\frac{\sqrt{3}-i}{\sqrt{3}+i}\right)^{100}$ を計算せよ。[15 点]

解答 $\frac{\sqrt{3}-i}{\sqrt{3}+i}=\frac{(\sqrt{3}-i)^2}{(\sqrt{3})^2-i^2}=\frac{2-2\sqrt{3}i}{4}=\frac{1}{2}-\frac{\sqrt{3}}{2}i=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)$

$\alpha=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)$ とおくと　$\alpha^6=\cos(-2\pi)+i\sin(-2\pi)=1$

よって　$\left(\frac{\sqrt{3}-i}{\sqrt{3}+i}\right)^{100}=\alpha^{100}=\alpha^{6\times16+4}=(\alpha^6)^{16}\times\alpha^4=\alpha^4$

$$=\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}^4$$
$$=\cos\left(-\frac{4}{3}\pi\right)+i\sin\left(-\frac{4}{3}\pi\right)=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

解説

$\frac{\sqrt{3}-i}{\sqrt{3}+i}=\frac{(\sqrt{3}-i)^2}{(\sqrt{3})^2-i^2}=\frac{2-2\sqrt{3}i}{4}=\frac{1}{2}-\frac{\sqrt{3}}{2}i=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)$

$\alpha=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)$ とおくと　$\alpha^6=\cos(-2\pi)+i\sin(-2\pi)=1$

よって　$\left(\frac{\sqrt{3}-i}{\sqrt{3}+i}\right)^{100}=\alpha^{100}=\alpha^{6\times16+4}=(\alpha^6)^{16}\times\alpha^4=\alpha^4$

$$=\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}^4=\cos\left(-\frac{4}{3}\pi\right)+i\sin\left(-\frac{4}{3}\pi\right)$$
$$=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

$\boxed{8}$ 次の値を求めよ。

(1) $\left(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi\right)^5$　(2) $(1-i)^8$　(3) $(1+\sqrt{3}i)^{-7}$

解答 (1) $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$　(2) 16　(3) $\frac{1-\sqrt{3}i}{256}$

解説

(1) $\left(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi\right)^5=\cos\left(5\times\frac{2}{3}\pi\right)+i\sin\left(5\times\frac{2}{3}\pi\right)=\cos\frac{10}{3}\pi+i\sin\frac{10}{3}\pi$

$=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

(2) $1-i=\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=\sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right\}$

よって　$(1-i)^8=\left[\sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right\}\right]^8$

$=(\sqrt{2})^8\left[\cos\left\{8\times\left(-\frac{\pi}{4}\right)\right\}+i\sin\left\{8\times\left(-\frac{\pi}{4}\right)\right\}\right]$

$=2^4\{\cos(-2\pi)+i\sin(-2\pi)\}=16\times1=16$

(3) $1+\sqrt{3}i=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

よって　$(1+\sqrt{3}i)^{-7}$

$=\left\{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right\}^{-7}$

$=2^{-7}\left[\cos\left\{(-7)\times\frac{\pi}{3}\right\}+i\sin\left\{(-7)\times\frac{\pi}{3}\right\}\right]$

$=2^{-7}\left\{\cos\left(-\frac{7}{3}\pi\right)+i\sin\left(-\frac{7}{3}\pi\right)\right\}$

$=\frac{1}{128}\times\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=\frac{1-\sqrt{3}i}{256}$

9 次の値を求めよ。

(1) $\left\{2\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)\right\}^6$　(2) $(\sqrt{3}+i)^5$　(3) $\left(\frac{1-i}{\sqrt{2}}\right)^{-7}$

解答　(1) $64i$　(2) $-16\sqrt{3}+16i$　(3) $\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$

解説

(1) $\left\{2\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)\right\}^6=2^6\left\{\cos\left(6\times\frac{\pi}{12}\right)+i\sin\left(6\times\frac{\pi}{12}\right)\right\}$

$=2^6\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=64i$

(2) $\sqrt{3}+i=2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

よって　$(\sqrt{3}+i)^5=\left\{2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right\}^5=2^5\left\{\cos\left(5\times\frac{\pi}{6}\right)+i\sin\left(5\times\frac{\pi}{6}\right)\right\}$

$=2^5\left(\cos\frac{5}{6}\pi+i\sin\frac{5}{6}\pi\right)=2^5\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$

$=-16\sqrt{3}+16i$

(3) $1-i=\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=\sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right\}$

よって　$\left(\frac{1-i}{\sqrt{2}}\right)^{-7}=\left\{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right\}^{-7}$

$=\cos\left\{(-7)\times\left(-\frac{\pi}{4}\right)\right\}+i\sin\left\{(-7)\times\left(-\frac{\pi}{4}\right)\right\}$

$=\cos\frac{7}{4}\pi+i\sin\frac{7}{4}\pi=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$

10 次の式を計算せよ。

(1) $\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^4$　(2) $\left(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi\right)^{-5}$

(3) $\left\{2\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)\right\}^6$　(4) $\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^3$

解答　(1) -1　(2) $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$　(3) $32+32\sqrt{3}i$　(4) $-i$

解説

(1) 与式$=\cos\pi+i\sin\pi=-1$

(2) 与式$=\cos\left(-\frac{10}{3}\pi\right)+i\sin\left(-\frac{10}{3}\pi\right)$

$=\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

(3) 与式$=2^6\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)^6=64\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

$=64\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=32+32\sqrt{3}i$

(4) 与式$=\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}^3$

$=\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)=-i$

11 次の式を計算せよ。

(1) $(1+i)^{12}$　(2) $\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-5}$　(3) $\left(\frac{3-\sqrt{3}i}{2}\right)^8$　(4) $(-\sqrt{3}+i)^{-4}$

解答　(1) -64　(2) $\frac{1}{2}-\frac{\sqrt{3}}{2}i$　(3) $-\frac{81}{2}+\frac{81\sqrt{3}}{2}i$　(4) $-\frac{1}{32}+\frac{\sqrt{3}}{32}i$

解説

(1) $1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ であるから

与式$=(\sqrt{2})^{12}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{12}=(\sqrt{2})^{12}(\cos3\pi+i\sin3\pi)$

$=64(\cos\pi+i\sin\pi)=-64$

(2) $\frac{1}{2}-\frac{\sqrt{3}}{2}i=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)$ であるから

与式$=\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}^{-5}$

$=\cos\frac{5}{3}\pi+i\sin\frac{5}{3}\pi=\frac{1}{2}-\frac{\sqrt{3}}{2}i$

(3) $\frac{3-\sqrt{3}i}{2}=\sqrt{3}\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}$ であるから

与式$=(\sqrt{3})^8\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}^8=(\sqrt{3})^8\left\{\cos\left(-\frac{4}{3}\pi\right)+i\sin\left(-\frac{4}{3}\pi\right)\right\}$

$=81\left(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi\right)=-\frac{81}{2}+\frac{81\sqrt{3}}{2}i$

(4) $-\sqrt{3}+i=2\left(\cos\frac{5}{6}\pi+i\sin\frac{5}{6}\pi\right)$ であるから

与式$=2^{-4}\left(\cos\frac{5}{6}\pi+i\sin\frac{5}{6}\pi\right)^{-4}=2^{-4}\left\{\cos\left(-\frac{10}{3}\pi\right)+i\sin\left(-\frac{10}{3}\pi\right)\right\}$

$=\frac{1}{16}\left(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi\right)$

$=\frac{1}{16}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=-\frac{1}{32}+\frac{\sqrt{3}}{32}i$

12 次の式を計算せよ。

(1) $\frac{1}{(1-i)^9}$　(2) $\left(\frac{\sqrt{3}-i}{\sqrt{3}+i}\right)^3$　(3) $\left(\frac{-1+i}{1+\sqrt{3}i}\right)^8$　(4) $\left(\frac{5-i}{2-3i}\right)^{10}$

解答　(1) $\frac{1}{32}+\frac{1}{32}i$　(2) -1　(3) $-\frac{1}{32}-\frac{\sqrt{3}}{32}i$　(4) $32i$

解説

(1) $\frac{1}{(1-i)^9}=(1-i)^{-9}$

$1-i=\sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right\}$ であるから

与式$=(\sqrt{2})^{-9}\left\{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right\}^{-9}=\frac{1}{(\sqrt{2})^9}\left(\cos\frac{9}{4}\pi+i\sin\frac{9}{4}\pi\right)$

$=\frac{1}{16\sqrt{2}}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\frac{1}{16\sqrt{2}}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=\frac{1}{32}+\frac{1}{32}i$

(2) $\frac{\sqrt{3}-i}{\sqrt{3}+i}=\frac{2\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}}{2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)$

よって　与式$=\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}^3=\cos(-\pi)+i\sin(-\pi)=-1$

(3) $\frac{-1+i}{1+\sqrt{3}i}=\frac{\sqrt{2}\left(\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi\right)}{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}=\frac{1}{\sqrt{2}}\left(\cos\frac{5}{12}\pi+i\sin\frac{5}{12}\pi\right)$

よって　与式$=\frac{1}{(\sqrt{2})^8}\left(\cos\frac{5}{12}\pi+i\sin\frac{5}{12}\pi\right)^8=\frac{1}{16}\left(\cos\frac{10}{3}\pi+i\sin\frac{10}{3}\pi\right)$

$=\frac{1}{16}\left\{\cos\left(-\frac{2}{3}\pi\right)+i\sin\left(-\frac{2}{3}\pi\right)\right\}$

$=\frac{1}{16}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=-\frac{1}{32}-\frac{\sqrt{3}}{32}i$

(4) $\frac{5-i}{2-3i}=\frac{(5-i)(2+3i)}{(2-3i)(2+3i)}=\frac{10+15i-2i-3i^2}{2^2+3^2}$

$=\frac{13+13i}{13}=1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

よって　与式$=(\sqrt{2})^{10}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{10}=32\left(\cos\frac{5}{2}\pi+i\sin\frac{5}{2}\pi\right)$

$=32\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=32i$

13 $\left(\frac{2+\sqrt{3}-i}{2+\sqrt{3}+i}\right)^{1000}$ を計算せよ。

解答　$-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

解説

$\frac{2+\sqrt{3}-i}{2+\sqrt{3}+i}=\frac{(2+\sqrt{3}-i)^2}{(2+\sqrt{3})^2-i^2}=\frac{(2+\sqrt{3})^2-2(2+\sqrt{3})i+i^2}{8+4\sqrt{3}}$

$$=\frac{6+4\sqrt{3}-2(2+\sqrt{3})i}{8+4\sqrt{3}}=\frac{2\sqrt{3}(2+\sqrt{3})-2(2+\sqrt{3})i}{4(2+\sqrt{3})}$$

$$=\frac{\sqrt{3}}{2}-\frac{1}{2}i=\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)$$

$\alpha=\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)$ とおくと　$\alpha^{12}=\cos(-2\pi)+i\sin(-2\pi)=1$

よって　$\left(\frac{2+\sqrt{3}-i}{2+\sqrt{3}+i}\right)^{1000}=\alpha^{1000}=\alpha^{12\times83+4}=(\alpha^{12})^{83}\times\alpha^4=\alpha^4$

$$=\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}^4$$

$$=\cos\left(-\frac{2}{3}\pi\right)+i\sin\left(-\frac{2}{3}\pi\right)=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

14 次の式を計算せよ。

(1) $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^7$　　(2) $\left(\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi\right)^{-6}$

(3) $\left\{\sqrt{2}\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)\right\}^{12}$　　(4) $\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)^{10}$

解答　(1) $-\frac{\sqrt{3}}{2}-\frac{1}{2}i$　(2) 1　(3) $-32+32\sqrt{3}i$　(4) $-i$

解説

(1) $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^7=\cos\frac{7}{6}\pi+i\sin\frac{7}{6}\pi=-\frac{\sqrt{3}}{2}-\frac{1}{2}i$

(2) $\left(\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi\right)^{-6}=\cos(-8\pi)+i\sin(-8\pi)=1$

(3) $\left\{\sqrt{2}\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)\right\}^{12}=(\sqrt{2})^{12}\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)^{12}$

$$=2^6\left(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi\right)$$

$$=64\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=-32+32\sqrt{3}i$$

(4) $\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)^{10}=\left\{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right\}^{10}$

$$=\cos\left(-\frac{5}{2}\pi\right)+i\sin\left(-\frac{5}{2}\pi\right)=-i$$

15 次の式を計算せよ。

(1) $(1+\sqrt{3}i)^6$　　(2) $\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)^7$

(3) $(1-i)^{-10}$　　(4) $\left(\frac{3+\sqrt{3}i}{2}\right)^{-4}$

解答　(1) 64　(2) $-\frac{\sqrt{3}}{2}+\frac{1}{2}i$　(3) $\frac{1}{32}i$　(4) $-\frac{1}{18}-\frac{\sqrt{3}}{18}i$

解説

(1) $1+\sqrt{3}i=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ であるから

$$(1+\sqrt{3}i)^6=2^6\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^6=2^6(\cos2\pi+i\sin2\pi)=64$$

(2) $\frac{\sqrt{3}}{2}-\frac{1}{2}i=\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)$ であるから

$$\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)^7=\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}^7$$

$$=\cos\left(-\frac{7}{6}\pi\right)+i\sin\left(-\frac{7}{6}\pi\right)=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

(3) $1-i=\sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right\}$ であるから

$$(1-i)^{-10}=(\sqrt{2})^{-10}\left\{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right\}^{-10}$$

$$=\frac{1}{2^5}\left(\cos\frac{5}{2}\pi+i\sin\frac{5}{2}\pi\right)=\frac{1}{32}i$$

(4) $\frac{3+\sqrt{3}i}{2}=\sqrt{3}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ であるから

$$\left(\frac{3+\sqrt{3}i}{2}\right)^{-4}=(\sqrt{3})^{-4}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^{-4}$$

$$=\frac{1}{3^2}\left\{\cos\left(-\frac{2}{3}\pi\right)+i\sin\left(-\frac{2}{3}\pi\right)\right\}$$

$$=\frac{1}{9}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=-\frac{1}{18}-\frac{\sqrt{3}}{18}i$$

16 $\left(\frac{1-\sqrt{3}i}{\sqrt{2}}\right)^{10}$ を計算せよ。

解答　$-16+16\sqrt{3}i$

解説

$1-\sqrt{3}i=2\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}$ であるから

$$\frac{1-\sqrt{3}i}{\sqrt{2}}=\sqrt{2}\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}$$

よって　$\left(\frac{1-\sqrt{3}i}{\sqrt{2}}\right)^{10}=(\sqrt{2})^{10}\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}^{10}$

$$=2^5\left\{\cos\left(-\frac{10}{3}\pi\right)+i\sin\left(-\frac{10}{3}\pi\right)\right\}$$

$$=32\left(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi\right)$$

$$=32\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=-16+16\sqrt{3}i$$

17 次の式を計算せよ。

(1) $\left(\frac{1+i}{1-\sqrt{3}i}\right)^6$　　(2) $\left(\frac{1+4i}{3-5i}\right)^{-7}$

解答　(1) $-\frac{1}{8}i$　(2) $-8+8i$

解説

(1) $1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$1-\sqrt{3}i=2\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}$

よって　$\frac{1+i}{1-\sqrt{3}i}=\frac{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}{2\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}}$

$$=\frac{1}{\sqrt{2}}\left\{\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)\right\}$$

$$=\frac{1}{\sqrt{2}}\left(\cos\frac{7}{12}\pi+i\sin\frac{7}{12}\pi\right)$$

したがって　与式$=\left(\frac{1}{\sqrt{2}}\right)^6\left(\cos\frac{7}{12}\pi+i\sin\frac{7}{12}\pi\right)^6$

$$=\frac{1}{8}\left(\cos\frac{7}{2}\pi+i\sin\frac{7}{2}\pi\right)=-\frac{1}{8}i$$

(2) $\frac{1+4i}{3-5i}=\frac{(1+4i)(3+5i)}{(3-5i)(3+5i)}=\frac{3+5i+12i+20i^2}{3^2+5^2}$

$$=\frac{-17+17i}{34}=\frac{-1+i}{2}$$

$$=\frac{1}{\sqrt{2}}\left(\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi\right)$$

よって　与式$=\left(\frac{1}{\sqrt{2}}\right)^{-7}\left(\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi\right)^{-7}$

$$=(\sqrt{2})^7\left\{\cos\left(-\frac{21}{4}\pi\right)+i\sin\left(-\frac{21}{4}\pi\right)\right\}$$

$$=8\sqrt{2}\left(\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi\right)$$

$$=8\sqrt{2}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=-8+8i$$

18 (1) $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^9=\cos\left(9\cdot\frac{\pi}{3}\right)+i\sin\left(9\cdot\frac{\pi}{3}\right)$

$$=\cos3\pi+i\sin3\pi=-1$$

(2) $\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{-6}=\cos\left(-6\cdot\frac{\pi}{4}\right)+i\sin\left(-6\cdot\frac{\pi}{4}\right)$

$$=\cos\left(-\frac{3}{2}\pi\right)+i\sin\left(-\frac{3}{2}\pi\right)=i$$

解説

19 次の計算をせよ。

(1) $\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^8$　　(2) $\left(\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}\right)^{-12}$

解答　(1) 1　(2) $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

解説

(1) $\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^8=\cos\left(8\cdot\frac{\pi}{4}\right)+i\sin\left(8\cdot\frac{\pi}{4}\right)=\cos2\pi+i\sin2\pi=1$

(2) $\left(\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}\right)^{-12}=\cos\left(-12\cdot\frac{\pi}{9}\right)+i\sin\left(-12\cdot\frac{\pi}{9}\right)$

$$=\cos\left(-\frac{4}{3}\pi\right)+i\sin\left(-\frac{4}{3}\pi\right)=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

20 $(\sqrt{3}+i)^6$ を計算せよ。

解答　-64

解説

$\sqrt{3}+i$ を極形式で表すと

$$\sqrt{3}+i=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$

よって，ド・モアブルの定理により

$$\begin{aligned}(\sqrt{3}+i)^6&=2^6\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^6\\&=2^6\left\{\cos\left(6\cdot\frac{\pi}{6}\right)+i\sin\left(6\cdot\frac{\pi}{6}\right)\right\}\\&=2^6(\cos\pi+i\sin\pi)\\&=64\cdot(-1)=-64\end{aligned}$$

$\boxed{21}$ 次の計算をせよ。

(1) $(1+\sqrt{3}\,i)^3$　　(2) $\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)^8$

(3) $(\sqrt{3}-i)^4$　　(4) $(1-i)^{-8}$

解答 (1) -8　(2) 1　(3) $-8-8\sqrt{3}\,i$　(4) $\frac{1}{16}$

解説

(1) $1+\sqrt{3}\,i$ を極形式で表すと　$1+\sqrt{3}\,i=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

よって，ド・モアブルの定理により

$$\begin{aligned}(1+\sqrt{3}\,i)^3&=2^3\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^3=2^3\left\{\cos\left(3\cdot\frac{\pi}{3}\right)+i\sin\left(3\cdot\frac{\pi}{3}\right)\right\}\\&=2^3(\cos\pi+i\sin\pi)=8\cdot(-1)=-8\end{aligned}$$

(2) $-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$ を極形式で表すと　$-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i=\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi$

よって，ド・モアブルの定理により

$$\begin{aligned}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)^8&=\left(\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi\right)^8=\cos\left(8\cdot\frac{3}{4}\pi\right)+i\sin\left(8\cdot\frac{3}{4}\pi\right)\\&=\cos 6\pi+i\sin 6\pi=1\end{aligned}$$

(3) $\sqrt{3}-i$ を極形式で表すと　$\sqrt{3}-i=2\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}$

よって，ド・モアブルの定理により

$$\begin{aligned}(\sqrt{3}-i)^4&=2^4\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}^4\\&=2^4\left\{\cos\left(4\cdot\left(-\frac{\pi}{6}\right)\right)+i\sin\left(4\cdot\left(-\frac{\pi}{6}\right)\right)\right\}\\&=2^4\left\{\cos\left(-\frac{2}{3}\pi\right)+i\sin\left(-\frac{2}{3}\pi\right)\right\}\\&=16\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=-8-8\sqrt{3}\,i\end{aligned}$$

(4) $1-i$ を極形式で表すと　$1-i=\sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right\}$

よって，ド・モアブルの定理により

$$\begin{aligned}(1-i)^{-8}&=(\sqrt{2})^{-8}\left\{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right\}^{-8}\\&=\frac{1}{16}\left\{\cos\left(-8\cdot\left(-\frac{\pi}{4}\right)\right)+i\sin\left(-8\cdot\left(-\frac{\pi}{4}\right)\right)\right\}\\&=\frac{1}{16}(\cos 2\pi+i\sin 2\pi)=\frac{1}{16}(1+0i)=\frac{1}{16}\end{aligned}$$

$\boxed{22}$ 次の計算をせよ。

(1) $(-1+\sqrt{3}\,i)^6$　　(2) $(1-i)^9$

解答 (1) 64　(2) $16-16i$

解説

(1) $-1+\sqrt{3}\,i$ を極形式で表すと

$$-1+\sqrt{3}\,i=2\left(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi\right)$$

よって，ド・モアブルの定理により

$$\begin{aligned}(-1+\sqrt{3}\,i)^6&=2^6\left(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi\right)^6\\&=2^6\left\{\cos\left(6\cdot\frac{2}{3}\pi\right)+i\sin\left(6\cdot\frac{2}{3}\pi\right)\right\}\\&=2^6(\cos 4\pi+i\sin 4\pi)=64\cdot 1=64\end{aligned}$$

(2) $1-i$ を極形式で表すと

$$1-i=\sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right\}$$

よって，ド・モアブルの定理により

$$\begin{aligned}(1-i)^9&=(\sqrt{2})^9\left\{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right\}^9\\&=16\sqrt{2}\left\{\cos\left(9\cdot\left(-\frac{\pi}{4}\right)\right)+i\sin\left(9\cdot\left(-\frac{\pi}{4}\right)\right)\right\}\\&=16\sqrt{2}\left\{\cos\left(-\frac{9}{4}\pi\right)+i\sin\left(-\frac{9}{4}\pi\right)\right\}\\&=16\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=16-16i\end{aligned}$$

$\boxed{23}$ 次の計算をせよ。

(1) $\left(\frac{1-i}{1+\sqrt{3}\,i}\right)^6$　　(2) $\left\{\left(\frac{\sqrt{3}+i}{2}\right)^8+\left(\frac{\sqrt{3}-i}{2}\right)^8\right\}^2$

解答 (1) $\frac{i}{8}$　(2) 1

解説

(1) $1-i$，$1+\sqrt{3}\,i$ を極形式で表すと

$$1-i=\sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right\}$$

$$1+\sqrt{3}\,i=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

よって

$$\begin{aligned}\frac{1-i}{1+\sqrt{3}\,i}&=\frac{\sqrt{2}}{2}\left\{\cos\left(-\frac{\pi}{4}-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{4}-\frac{\pi}{3}\right)\right\}\\&=\frac{1}{\sqrt{2}}\left\{\cos\left(-\frac{7}{12}\pi\right)+i\sin\left(-\frac{7}{12}\pi\right)\right\}\end{aligned}$$

したがって，ド・モアブルの定理により

$$\begin{aligned}\left(\frac{1-i}{1+\sqrt{3}\,i}\right)^6&=\left(\frac{1}{\sqrt{2}}\right)^6\left\{\cos\left(-\frac{7}{12}\pi\right)+i\sin\left(-\frac{7}{12}\pi\right)\right\}^6\\&=\frac{1}{8}\left\{\cos\left(6\cdot\left(-\frac{7}{12}\pi\right)\right)+i\sin\left(6\cdot\left(-\frac{7}{12}\pi\right)\right)\right\}\\&=\frac{1}{8}\left\{\cos\left(-\frac{7}{2}\pi\right)+i\sin\left(-\frac{7}{2}\pi\right)\right\}\\&=\frac{1}{8}(0+i)=\frac{i}{8}\end{aligned}$$

(2) $\frac{\sqrt{3}+i}{2}$，$\frac{\sqrt{3}-i}{2}$ を極形式で表すと

$$\frac{\sqrt{3}+i}{2}=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6},\quad \frac{\sqrt{3}-i}{2}=\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)$$

よって，ド・モアブルの定理により

$$\begin{aligned}\left(\frac{\sqrt{3}+i}{2}\right)^8&=\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^8=\cos\left(8\cdot\frac{\pi}{6}\right)+i\sin\left(8\cdot\frac{\pi}{6}\right)\\&=\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi=-\frac{1}{2}-\frac{\sqrt{3}}{2}i\end{aligned}$$

$$\begin{aligned}\left(\frac{\sqrt{3}-i}{2}\right)^8&=\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}^8\\&=\cos\left(8\cdot\left(-\frac{\pi}{6}\right)\right)+i\sin\left(8\cdot\left(-\frac{\pi}{6}\right)\right)\\&=\cos\left(-\frac{4}{3}\pi\right)+i\sin\left(-\frac{4}{3}\pi\right)=-\frac{1}{2}+\frac{\sqrt{3}}{2}i\end{aligned}$$

したがって

$$\begin{aligned}\left\{\left(\frac{\sqrt{3}+i}{2}\right)^8+\left(\frac{\sqrt{3}-i}{2}\right)^8\right\}^2&=\left\{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)+\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right\}^2\\&=(-1)^2=1\end{aligned}$$

$\boxed{24}$ 次の式を計算せよ。

$$\frac{2+\sqrt{3}-i}{2+\sqrt{3}+i}={}^{ア}\boxed{},\quad \left(\frac{2+\sqrt{3}-i}{2+\sqrt{3}+i}\right)^3={}^{イ}\boxed{},\quad \left(\frac{2+\sqrt{3}-i}{2+\sqrt{3}+i}\right)^{2020}={}^{ウ}\boxed{}$$

解答 (ア) $\frac{\sqrt{3}}{2}-\frac{1}{2}i$　(イ) $-i$　(ウ) $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

解説

(ア)

$$\begin{aligned}\frac{2+\sqrt{3}-i}{2+\sqrt{3}+i}&=\frac{(2+\sqrt{3}-i)^2}{(2+\sqrt{3}+i)(2+\sqrt{3}-i)}=\frac{(2+\sqrt{3})^2-2(2+\sqrt{3})i-1}{(2+\sqrt{3})^2+1}\\&=\frac{6+4\sqrt{3}-2(2+\sqrt{3})i}{8+4\sqrt{3}}=\frac{2\sqrt{3}(\sqrt{3}+2)-2(2+\sqrt{3})i}{4(2+\sqrt{3})}\\&=\frac{\sqrt{3}}{2}-\frac{1}{2}i\end{aligned}$$

(イ) $\frac{2+\sqrt{3}-i}{2+\sqrt{3}+i}=\alpha$ とおくと，(ア) から　$\alpha=\frac{\sqrt{3}}{2}-\frac{1}{2}i$

α を極形式で表すと　$\alpha=\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)$

よって

$$\begin{aligned}\alpha^3&=\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}^3\\&=\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)=-i\end{aligned}$$

(ウ) $\alpha^{12}=\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}^{12}=\cos(-2\pi)+i\sin(-2\pi)=1$

よって

$$\begin{aligned}\alpha^{2020}&=\alpha^{12\times168+4}=(\alpha^{12})^{168}\cdot\alpha^4=\alpha^4\\&=\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}^4\\&=\cos\left(-\frac{2}{3}\pi\right)+i\sin\left(-\frac{2}{3}\pi\right)=-\frac{1}{2}-\frac{\sqrt{3}}{2}i\end{aligned}$$

$\boxed{25}$ 次の式を計算せよ。

(1) $\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)^6$　(2) $\left(\frac{1+i}{2}\right)^{15}$　(3) $(\sqrt{6}-\sqrt{2}\,i)^{-6}$

(4) $\left(\frac{1+\sqrt{3}\,i}{1+i}\right)^{12}$　(5) $(\sqrt{3}+i)^{10}+(\sqrt{3}-i)^{10}$

解答　(1) i　(2) $\frac{1}{256}-\frac{1}{256}i$　(3) $-\frac{1}{512}$　(4) -64　(5) 1024

解説

(1) $\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)^6=\cos\left(6\times\frac{\pi}{12}\right)+i\sin\left(6\times\frac{\pi}{12}\right)$

$=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=i$

(2) $1+i=\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ であるから

$\left(\frac{1+i}{2}\right)^{15}=\left\{\frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right\}^{15}$

$=\left(\frac{\sqrt{2}}{2}\right)^{15}\left\{\cos\left(15\times\frac{\pi}{4}\right)+i\sin\left(15\times\frac{\pi}{4}\right)\right\}$

$=\frac{1}{(\sqrt{2})^{15}}\left\{\cos\left(-\frac{\pi}{4}+4\pi\right)+i\sin\left(-\frac{\pi}{4}+4\pi\right)\right\}$

$=\frac{1}{(\sqrt{2})^{15}}\left\{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right\}$

$=\frac{1}{(\sqrt{2})^{15}}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=\frac{1-i}{(\sqrt{2})^{16}}=\frac{1}{256}-\frac{1}{256}i$

(3) $\sqrt{6}-\sqrt{2}\,i=2\sqrt{2}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=2\sqrt{2}\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}$

したがって　$(\sqrt{6}-\sqrt{2}\,i)^{-6}=(2\sqrt{2})^{-6}\left[\cos\left\{-6\times\left(-\frac{\pi}{6}\right)\right\}+i\sin\left\{-6\times\left(-\frac{\pi}{6}\right)\right\}\right]$

$=(2\sqrt{2})^{-6}(\cos\pi+i\sin\pi)$

$=-\frac{1}{(2\sqrt{2})^6}=-\frac{1}{2^9}=-\frac{1}{512}$

(4) $1+\sqrt{3}\,i=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$, $1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ であるから

$\frac{1+\sqrt{3}\,i}{1+i}=\frac{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}=\sqrt{2}\left\{\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\right\}$

$=\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$

よって　$\left(\frac{1+\sqrt{3}\,i}{1+i}\right)^{12}=\left\{\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)\right\}^{12}$

$=(\sqrt{2})^{12}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)^{12}$

$=64\left\{\cos\left(12\times\frac{\pi}{12}\right)+i\sin\left(12\times\frac{\pi}{12}\right)\right\}$

$=64(\cos\pi+i\sin\pi)=-64$

(5) $\sqrt{3}+i=2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ であるから

$(\sqrt{3}+i)^{10}=\left\{2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right\}^{10}=2^{10}\left(\cos\frac{5}{3}\pi+i\sin\frac{5}{3}\pi\right)$

$=2^{10}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=2^9(1-\sqrt{3}\,i)$

同様に　$(\sqrt{3}-i)^{10}=2^{10}\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}^{10}$

$=2^{10}\left\{\cos\left(-\frac{5}{3}\pi\right)+i\sin\left(-\frac{5}{3}\pi\right)\right\}$

$=2^{10}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=2^9(1+\sqrt{3}\,i)$

ゆえに　$(\sqrt{3}+i)^{10}+(\sqrt{3}-i)^{10}=2^9(1-\sqrt{3}\,i)+2^9(1+\sqrt{3}\,i)$

$=2^9\times2=2^{10}=1024$

$\boxed{26}$ 次の式を計算せよ。

(1) $\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)^{12}$　(2) $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^6$

(3) $\left(\cos\frac{5}{12}\pi+i\sin\frac{5}{12}\pi\right)^{-3}$　(4) $\left\{2\left(\cos\frac{\pi}{12}-i\sin\frac{\pi}{12}\right)\right\}^4$

解答　(1) $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$　(2) 1　(3) $-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$　(4) $8-8\sqrt{3}\,i$

解説

(1) $\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)^{12}=\cos\left(12\times\frac{\pi}{18}\right)+i\sin\left(12\times\frac{\pi}{18}\right)$

$=\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

(2) $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^6=\cos\left(6\times\frac{\pi}{3}\right)+i\sin\left(6\times\frac{\pi}{3}\right)$

$=\cos2\pi+i\sin2\pi=1$

(3) $\left(\cos\frac{5}{12}\pi+i\sin\frac{5}{12}\pi\right)^{-3}=\cos\left((-3)\times\frac{5}{12}\pi\right)+i\sin\left((-3)\times\frac{5}{12}\pi\right)$

$=\cos\left(-\frac{5}{4}\pi\right)+i\sin\left(-\frac{5}{4}\pi\right)=\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi$

$=-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$

(4) $\left\{2\left(\cos\frac{\pi}{12}-i\sin\frac{\pi}{12}\right)\right\}^4=2^4\left\{\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)\right\}^4$

$=16\left\{\cos\left(4\times\left(-\frac{\pi}{12}\right)\right)+i\sin\left(4\times\left(-\frac{\pi}{12}\right)\right)\right\}$

$=16\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}$

$=16\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=8-8\sqrt{3}\,i$

$\boxed{27}$ 次の式を計算せよ。

(1) $\frac{1}{(1+i)^6}$　(2) $\left(\frac{1+i}{\sqrt{3}+i}\right)^8$　(3) $\left(\frac{1-\sqrt{3}\,i}{1+\sqrt{3}\,i}\right)^{10}$　(4) $\left(\frac{5-i}{2-3i}\right)^8$

解答　(1) $\frac{1}{8}i$　(2) $-\frac{1}{32}+\frac{\sqrt{3}}{32}i$　(3) $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$　(4) 16

解説

(1) $1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

よって　$\frac{1}{(1+i)^6}=\left\{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right\}^{-6}$

$=(\sqrt{2})^{-6}\left\{\cos\left(-6\times\frac{\pi}{4}\right)+i\sin\left(-6\times\frac{\pi}{4}\right)\right\}$

$=\frac{1}{8}\left\{\cos\left(-\frac{3}{2}\pi\right)+i\sin\left(-\frac{3}{2}\pi\right)\right\}$

$=\frac{1}{8}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=\frac{1}{8}i$

(2) $\frac{1+i}{\sqrt{3}+i}=\frac{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}{2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}=\frac{1}{\sqrt{2}}\left\{\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\right\}$

$=\frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$

よって　$\left(\frac{1+i}{\sqrt{3}+i}\right)^8=\left\{\frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)\right\}^8$

$=\left(\frac{1}{\sqrt{2}}\right)^8\left\{\cos\left(8\times\frac{\pi}{12}\right)+i\sin\left(8\times\frac{\pi}{12}\right)\right\}$

$=\left(\frac{1}{2}\right)^4\left(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi\right)$

$=\frac{1}{16}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=-\frac{1}{32}+\frac{\sqrt{3}}{32}i$

(3) $\frac{1-\sqrt{3}\,i}{1+\sqrt{3}\,i}=\frac{2\left(\cos\frac{5}{3}\pi+i\sin\frac{5}{3}\pi\right)}{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}=\cos\left(\frac{5}{3}\pi-\frac{\pi}{3}\right)+i\sin\left(\frac{5}{3}\pi-\frac{\pi}{3}\right)$

$=\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi$

よって　$\left(\frac{1-\sqrt{3}\,i}{1+\sqrt{3}\,i}\right)^{10}=\left(\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi\right)^{10}$

$=\cos\left(10\times\frac{4}{3}\pi\right)+i\sin\left(10\times\frac{4}{3}\pi\right)$

$=\cos\frac{40}{3}\pi+i\sin\frac{40}{3}\pi=\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi$

$=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

(4) $\frac{5-i}{2-3i}=\frac{(5-i)(2+3i)}{(2-3i)(2+3i)}=\frac{10+13i-3i^2}{13}=\frac{13+13i}{13}=1+i$

$=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

よって　$\left(\frac{5-i}{2-3i}\right)^8=\left\{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right\}^8$

$=(\sqrt{2})^8\left\{\cos\left(8\times\frac{\pi}{4}\right)+i\sin\left(8\times\frac{\pi}{4}\right)\right\}$

$=16(\cos2\pi+i\sin2\pi)=16$

$\boxed{28}$ 次の式を計算せよ。

(1) $\left(\frac{\sqrt{3}-i}{3+\sqrt{3}\,i}\right)^4$　(2) $\left(\frac{-2+2\sqrt{3}\,i}{1+i}\right)^6$　(3) $\left(\frac{-1+7i}{4-3i}\right)^8$

解答　(1) $-\frac{1}{18}+\frac{\sqrt{3}}{18}i$　(2) $512i$　(3) 16

解説

(1) $\frac{\sqrt{3}-i}{3+\sqrt{3}\,i}=\frac{2\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}}{2\sqrt{3}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}$

$=\frac{2}{2\sqrt{3}}\left\{\cos\left(-\frac{\pi}{6}-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}-\frac{\pi}{6}\right)\right\}$

$$=\frac{1}{\sqrt{3}}\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}$$

よって　与式$=\left(\frac{1}{\sqrt{3}}\right)^4\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}^4$

$$=\frac{1}{9}\left\{\cos\left(-\frac{4}{3}\pi\right)+i\sin\left(-\frac{4}{3}\pi\right)\right\}$$

$$=\frac{1}{9}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=-\frac{1}{18}+\frac{\sqrt{3}}{18}i$$

(2) $\frac{-2+2\sqrt{3}\,i}{1+i}=\frac{4\left(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi\right)}{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}=\frac{4}{\sqrt{2}}\left\{\cos\left(\frac{2}{3}\pi-\frac{\pi}{4}\right)+i\sin\left(\frac{2}{3}\pi-\frac{\pi}{4}\right)\right\}$

$$=2\sqrt{2}\left(\cos\frac{5}{12}\pi+i\sin\frac{5}{12}\pi\right)$$

よって　与式$=(2\sqrt{2})^6\left(\cos\frac{5}{12}\pi+i\sin\frac{5}{12}\pi\right)^6=512\left(\cos\frac{5}{2}\pi+i\sin\frac{5}{2}\pi\right)=512i$

(3) $\frac{-1+7i}{4-3i}=\frac{(-1+7i)(4+3i)}{(4-3i)(4+3i)}=\frac{-4-3i+28i+21i^2}{4^2+3^2}$

$$=\frac{-25+25i}{25}=-1+i=\sqrt{2}\left(\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi\right)$$

よって　与式$=(\sqrt{2})^8\left(\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi\right)^8=16(\cos 6\pi+i\sin 6\pi)=16$

$\boxed{29}$ 次の計算をせよ。

(1) $\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)^5$　　(2) $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^{-4}$

(3) $\left\{2\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)\right\}^6$　　(4) $\left\{-\sqrt{2}\left(\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}\right)\right\}^5$

(5) $\left\{\cos\left(-\frac{2}{3}\pi\right)+i\sin\left(-\frac{2}{3}\pi\right)\right\}^{-3}$　　(6) $\left\{-2\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)\right\}^{-4}$

解答　(1) -1　(2) $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$　(3) $32+32\sqrt{3}\,i$　(4) $-4\sqrt{2}\,i$　(5) 1

(6) $\frac{1}{32}-\frac{\sqrt{3}}{32}i$

解説

(1) $\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)^5=\cos\pi+i\sin\pi=-1$

(2) $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^{-4}=\cos\left(-\frac{2}{3}\pi\right)+i\sin\left(-\frac{2}{3}\pi\right)=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

(3) $\left\{2\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)\right\}^6=2^6\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=64\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=32+32\sqrt{3}\,i$

(4) $\left\{-\sqrt{2}\left(\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}\right)\right\}^5=(-\sqrt{2})^5\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=-4\sqrt{2}\,i$

(5) $\left\{\cos\left(-\frac{2}{3}\pi\right)+i\sin\left(-\frac{2}{3}\pi\right)\right\}^{-3}=\cos 2\pi+i\sin 2\pi=1$

(6) $\left\{-2\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)\right\}^{-4}=(-2)^{-4}\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}$

$$=\frac{1}{16}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=\frac{1}{32}-\frac{\sqrt{3}}{32}i$$

$\boxed{30}$ 次の計算をせよ。

(1) $\frac{1}{(1-i)^9}$　　(2) $\left(\frac{5-i}{2-3i}\right)^{10}$　　(3) $\frac{(1+i)^6}{(1-\sqrt{3}\,i)^3}$

(4) $\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^{10}+\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)^{10}$　　(5) $\left(\sin\frac{2}{5}\pi+i\cos\frac{2}{5}\pi\right)^{10}$

解答　(1) $\frac{1}{32}+\frac{1}{32}i$　(2) $32i$　(3) i　(4) 1　(5) -1

解説

(1) $\frac{1}{(1-i)^9}=(1-i)^{-9}=\left[\sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right\}\right]^{-9}$

$$=\frac{1}{(\sqrt{2})^9}\left(\cos\frac{9}{4}\pi+i\sin\frac{9}{4}\pi\right)=\frac{1}{16\sqrt{2}}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

$$=\frac{1}{16\sqrt{2}}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=\frac{1}{32}+\frac{1}{32}i$$

(2) $\frac{5-i}{2-3i}=\frac{(5-i)(2+3i)}{(2-3i)(2+3i)}=\frac{10+15i-2i-3i^2}{4-9i^2}$

$$=\frac{13+13i}{13}=1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

よって

$$\left(\frac{5-i}{2-3i}\right)^{10}=\left\{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right\}^{10}=(\sqrt{2})^{10}\left(\cos\frac{5}{2}\pi+i\sin\frac{5}{2}\pi\right)$$

$$=32\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=32i$$

(3) $(1+i)^6=\left\{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right\}^6=(\sqrt{2})^6\left(\cos\frac{3}{2}\pi+i\sin\frac{3}{2}\pi\right)$

$$=8\left(\cos\frac{3}{2}\pi+i\sin\frac{3}{2}\pi\right)=-8i$$

$$(1-\sqrt{3}\,i)^3=\left[2\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}\right]^3=2^3\{\cos(-\pi)+i\sin(-\pi)\}$$

$$=8\{\cos(-\pi)+i\sin(-\pi)\}=-8$$

よって　$\frac{(1+i)^6}{(1-\sqrt{3}\,i)^3}=\frac{-8i}{-8}=i$

(4) $\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^{10}=\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^{10}=\cos\frac{5}{3}\pi+i\sin\frac{5}{3}\pi$

$$\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)^{10}=\left\{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right\}^{10}=\cos\left(-\frac{5}{3}\pi\right)+i\sin\left(-\frac{5}{3}\pi\right)$$

$$=\cos\frac{5}{3}\pi-i\sin\frac{5}{3}\pi$$

よって

$$\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^{10}+\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)^{10}=\left(\cos\frac{5}{3}\pi+i\sin\frac{5}{3}\pi\right)+\left(\cos\frac{5}{3}\pi-i\sin\frac{5}{3}\pi\right)$$

$$=2\cos\frac{5}{3}\pi=2\cdot\frac{1}{2}=1$$

(5) $\left(\sin\frac{2}{5}\pi+i\cos\frac{2}{5}\pi\right)^{10}=\left\{\cos\left(\frac{\pi}{2}-\frac{2}{5}\pi\right)+i\sin\left(\frac{\pi}{2}-\frac{2}{5}\pi\right)\right\}^{10}$

$$=\left(\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}\right)^{10}=\cos\pi+i\sin\pi=-1$$