部分分数分解クイズ

1 数列 $\dfrac{1}{2\cdot5}$，$\dfrac{1}{5\cdot8}$，$\dfrac{1}{8\cdot11}$，…… の初項から第 n 項までの和を求めよ。

解答 $\dfrac{n}{2(3n+2)}$

解説

この数列の第 k 項は $\dfrac{1}{(3k-1)(3k+2)}$

$$\frac{1}{(3k-1)(3k+2)}=\frac{1}{3}\cdot\frac{(3k+2)-(3k-1)}{(3k-1)(3k+2)}=\frac{1}{3}\left(\frac{1}{3k-1}-\frac{1}{3k+2}\right)$$

この式に $k=1$，2，……，n を代入して，辺々を加えると

$$\frac{1}{2\cdot5}+\frac{1}{5\cdot8}+\frac{1}{8\cdot11}+\cdots\cdots+\frac{1}{(3n-1)(3n+2)}$$

$$=\frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right)+\frac{1}{3}\left(\frac{1}{5}-\frac{1}{8}\right)+\frac{1}{3}\left(\frac{1}{8}-\frac{1}{11}\right)+\cdots\cdots+\frac{1}{3}\left(\frac{1}{3n-1}-\frac{1}{3n+2}\right)$$

$$=\frac{1}{3}\left\{\left(\frac{1}{2}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{5}}-\cancel{\frac{1}{8}}\right)+\left(\cancel{\frac{1}{8}}-\cancel{\frac{1}{11}}\right)+\cdots\cdots+\left(\cancel{\frac{1}{3n-1}}-\frac{1}{3n+2}\right)\right\}$$

$$=\frac{1}{3}\left(\frac{1}{2}-\frac{1}{3n+2}\right)=\frac{1}{3}\cdot\frac{3n+2-2}{2(3n+2)}=\frac{n}{2(3n+2)}$$

2 次の数列の初項から第 n 項までの和を求めよ。

(1) $\dfrac{2}{1\cdot3}$，$\dfrac{2}{3\cdot5}$，$\dfrac{2}{5\cdot7}$，……　　(2) $\dfrac{1}{1\cdot5}$，$\dfrac{1}{5\cdot9}$，$\dfrac{1}{9\cdot13}$，……

解答 (1) $\dfrac{2n}{2n+1}$　(2) $\dfrac{n}{4n+1}$

解説

(1) この数列の第 k 項は $\dfrac{2}{(2k-1)(2k+1)}=\dfrac{(2k+1)-(2k-1)}{(2k-1)(2k+1)}=\dfrac{1}{2k-1}-\dfrac{1}{2k+1}$

ゆえに，初項から第 n 項までの和は

$$\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$$

$$=1-\frac{1}{2n+1}=\frac{2n}{2n+1}$$

(2) この数列の第 k 項は

$$\frac{1}{(4k-3)(4k+1)}=\frac{1}{4}\cdot\frac{(4k+1)-(4k-3)}{(4k-3)(4k+1)}=\frac{1}{4}\left(\frac{1}{4k-3}-\frac{1}{4k+1}\right)$$

ゆえに，初項から第 n 項までの和は

$$\frac{1}{4}\left[\left(\frac{1}{1}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{13}\right)+\cdots\cdots+\left(\frac{1}{4n-3}-\frac{1}{4n+1}\right)\right]$$

$$=\frac{1}{4}\left(1-\frac{1}{4n+1}\right)=\frac{n}{4n+1}$$

3 次の和を求めよ。ただし，(2) では $n\geqq2$ とする。

(1) $\displaystyle\sum_{k=1}^{n}\frac{1}{\sqrt{k+2}+\sqrt{k+1}}$　　(2) $\displaystyle\sum_{k=1}^{n}\frac{2}{(k+1)(k+3)}$

解答 (1) $\sqrt{n+2}-\sqrt{2}$　(2) $\dfrac{n(5n+13)}{6(n+2)(n+3)}$

解説

(1) $$\frac{1}{\sqrt{k+2}+\sqrt{k+1}}=\frac{\sqrt{k+2}-\sqrt{k+1}}{(\sqrt{k+2}+\sqrt{k+1})(\sqrt{k+2}-\sqrt{k+1})}$$

$$=\frac{\sqrt{k+2}-\sqrt{k+1}}{(k+2)-(k+1)}=\sqrt{k+2}-\sqrt{k+1}$$

であるから

$$\sum_{k=1}^{n}\frac{1}{\sqrt{k+2}+\sqrt{k+1}}=\sum_{k=1}^{n}(\sqrt{k+2}-\sqrt{k+1})$$

$$=(\sqrt{3}-\sqrt{2})+(\sqrt{4}-\sqrt{3})+(\sqrt{5}-\sqrt{4})$$

$$+\cdots\cdots+(\sqrt{n+1}-\sqrt{n})+(\sqrt{n+2}-\sqrt{n+1})$$

$$=\sqrt{n+2}-\sqrt{2}$$

(2) $\dfrac{2}{(k+1)(k+3)}=\dfrac{1}{k+1}-\dfrac{1}{k+3}$ であるから

$n\geqq2$ のとき

$$\sum_{k=1}^{n}\frac{2}{(k+1)(k+3)}=\sum_{k=1}^{n}\left(\frac{1}{k+1}-\frac{1}{k+3}\right)$$

$$=\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)$$

$$+\cdots\cdots+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)+\left(\frac{1}{n}-\frac{1}{n+2}\right)+\left(\frac{1}{n+1}-\frac{1}{n+3}\right)$$

$$=\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}=\frac{n(5n+13)}{6(n+2)(n+3)}$$

4 次の和を求めよ。ただし，(2) では $n\geqq2$ とする。

(1) $\displaystyle\sum_{k=1}^{n}\frac{1}{\sqrt{k+4}+\sqrt{k+3}}$　　(2) $\displaystyle\sum_{k=1}^{n}\frac{2}{(k+2)(k+4)}$

解答 (1) $\sqrt{n+4}-2$　(2) $\dfrac{n(7n+25)}{12(n+3)(n+4)}$

解説

(1) $$\frac{1}{\sqrt{k+4}+\sqrt{k+3}}=\frac{\sqrt{k+4}-\sqrt{k+3}}{(\sqrt{k+4}+\sqrt{k+3})(\sqrt{k+4}-\sqrt{k+3})}$$

$$=\frac{\sqrt{k+4}-\sqrt{k+3}}{(k+4)-(k+3)}=\sqrt{k+4}-\sqrt{k+3}$$

であるから $$\sum_{k=1}^{n}\frac{1}{\sqrt{k+4}+\sqrt{k+3}}=\sum_{k=1}^{n}(\sqrt{k+4}-\sqrt{k+3})$$

$$=(\sqrt{5}-\sqrt{4})+(\sqrt{6}-\sqrt{5})+(\sqrt{7}-\sqrt{6})$$

$$+\cdots\cdots+(\sqrt{n+3}-\sqrt{n+2})+(\sqrt{n+4}-\sqrt{n+3})$$

$$=\sqrt{n+4}-2$$

(2) $\dfrac{2}{(k+2)(k+4)}=\dfrac{1}{k+2}-\dfrac{1}{k+4}$ であるから

$n\geqq2$ のとき $$\sum_{k=1}^{n}\frac{2}{(k+2)(k+4)}=\sum_{k=1}^{n}\left(\frac{1}{k+2}-\frac{1}{k+4}\right)$$

$$=\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)$$

$$+\cdots\cdots+\left(\frac{1}{n}-\frac{1}{n+2}\right)+\left(\frac{1}{n+1}-\frac{1}{n+3}\right)+\left(\frac{1}{n+2}-\frac{1}{n+4}\right)$$

$$=\frac{1}{3}+\frac{1}{4}-\frac{1}{n+3}-\frac{1}{n+4}=\frac{n(7n+25)}{12(n+3)(n+4)}$$

5 数列 $\dfrac{1}{1\cdot2\cdot3}$，$\dfrac{1}{2\cdot3\cdot4}$，$\dfrac{1}{3\cdot4\cdot5}$，……，$\dfrac{1}{n(n+1)(n+2)}$ の和 S を求めよ。

解答 $\dfrac{n(n+3)}{4(n+1)(n+2)}$

解説

第 k 項は $$\frac{1}{k(k+1)(k+2)}=\frac{1}{2}\left\{\frac{1}{k(k+1)}-\frac{1}{(k+1)(k+2)}\right\}$$

よって $$S=\frac{1}{2}\left[\left(\frac{1}{1\cdot2}-\frac{1}{2\cdot3}\right)+\left(\frac{1}{2\cdot3}-\frac{1}{3\cdot4}\right)+\left(\frac{1}{3\cdot4}-\frac{1}{4\cdot5}\right)\right.$$

$$\left.+\cdots\cdots+\left\{\frac{1}{(n-1)n}-\frac{1}{n(n+1)}\right\}+\left\{\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\right\}\right]$$

$$=\frac{1}{2}\left\{\frac{1}{1\cdot2}-\frac{1}{(n+1)(n+2)}\right\}$$

$$=\frac{1}{2}\cdot\frac{(n+1)(n+2)-2}{2(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

6 数列 $\dfrac{1}{1\cdot4\cdot7}$，$\dfrac{1}{4\cdot7\cdot10}$，$\dfrac{1}{7\cdot10\cdot13}$，……，$\dfrac{1}{(3n-2)(3n+1)(3n+4)}$ の和 S を求めよ。

解答 $\dfrac{n(3n+5)}{8(3n+1)(3n+4)}$

解説

第 k 項は $$\frac{1}{(3k-2)(3k+1)(3k+4)}=\frac{1}{6}\left\{\frac{1}{(3k-2)(3k+1)}-\frac{1}{(3k+1)(3k+4)}\right\}$$

よって $$S=\frac{1}{6}\left[\left(\frac{1}{1\cdot4}-\frac{1}{4\cdot7}\right)+\left(\frac{1}{4\cdot7}-\frac{1}{7\cdot10}\right)+\left(\frac{1}{7\cdot10}-\frac{1}{10\cdot13}\right)\right.$$

$$+\cdots\cdots+\left\{\frac{1}{(3n-5)(3n-2)}-\frac{1}{(3n-2)(3n+1)}\right\}$$

$$\left.+\left\{\frac{1}{(3n-2)(3n+1)}-\frac{1}{(3n+1)(3n+4)}\right\}\right]$$

$$=\frac{1}{6}\left\{\frac{1}{4}-\frac{1}{(3n+1)(3n+4)}\right\}$$

$$=\frac{1}{6}\cdot\frac{(3n+1)(3n+4)-4}{4(3n+1)(3n+4)}=\frac{n(3n+5)}{8(3n+1)(3n+4)}$$

7 (1) 数列 $\dfrac{1}{1+\sqrt{3}}$，$\dfrac{1}{\sqrt{3}+\sqrt{5}}$，$\dfrac{1}{\sqrt{5}+\sqrt{7}}$，$\dfrac{1}{\sqrt{7}+3}$，…… の初項から第 n 項までの和を求めよ。

(2) 和 $\dfrac{1}{1\cdot3\cdot5}+\dfrac{1}{3\cdot5\cdot7}+\dfrac{1}{5\cdot7\cdot9}+\cdots\cdots+\dfrac{1}{(2n-1)(2n+1)(2n+3)}$ を求めよ。

解答 (1) $\dfrac{1}{2}(\sqrt{2n+1}-1)$　(2) $\dfrac{n(n+2)}{3(2n+1)(2n+3)}$

解説

(1) 第 k 項は

$$\frac{1}{\sqrt{2k-1}+\sqrt{2k+1}}=\frac{\sqrt{2k-1}-\sqrt{2k+1}}{(\sqrt{2k-1}+\sqrt{2k+1})(\sqrt{2k-1}-\sqrt{2k+1})}$$

$$=-\frac{1}{2}(\sqrt{2k-1}-\sqrt{2k+1})=\frac{1}{2}(\sqrt{2k+1}-\sqrt{2k-1})$$

ゆえに，初項から第 n 項までの和は

$$\frac{1}{2}\{(\sqrt{3}-1)+(\sqrt{5}-\sqrt{3})+(\sqrt{7}-\sqrt{5})+\cdots\cdots+(\sqrt{2n+1}-\sqrt{2n-1})\}$$
$$=\frac{1}{2}(\sqrt{2n+1}-1)$$

(2) 和の各項を，ある数列の初項，第 2 項，…… と考えると，その数列の第 k 項は

$$\frac{1}{(2k-1)(2k+1)(2k+3)}=\frac{1}{4}\cdot\frac{(2k+3)-(2k-1)}{(2k-1)(2k+1)(2k+3)}$$
$$=\frac{1}{4}\left\{\frac{1}{(2k-1)(2k+1)}-\frac{1}{(2k+1)(2k+3)}\right\}$$

よって，求める和は

$$\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)(2k+3)}$$
$$=\frac{1}{4}\sum_{k=1}^{n}\left\{\frac{1}{(2k-1)(2k+1)}-\frac{1}{(2k+1)(2k+3)}\right\}$$
$$=\frac{1}{4}\left[\left(\frac{1}{1\cdot3}-\frac{1}{3\cdot5}\right)+\left(\frac{1}{3\cdot5}-\frac{1}{5\cdot7}\right)+\left(\frac{1}{5\cdot7}-\frac{1}{7\cdot9}\right)\right.$$
$$\left.+\cdots\cdots+\left\{\frac{1}{(2n-1)(2n+1)}-\frac{1}{(2n+1)(2n+3)}\right\}\right]$$
$$=\frac{1}{4}\left\{\frac{1}{3}-\frac{1}{(2n+1)(2n+3)}\right\}=\frac{n(n+2)}{3(2n+1)(2n+3)}$$

8 恒等式 $\dfrac{1}{(3k-2)(3k+1)}=\dfrac{1}{3}\left(\dfrac{1}{3k-2}-\dfrac{1}{3k+1}\right)$ を利用して，次の和 S を求めよ。

$$S=\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+\cdots\cdots+\frac{1}{(3n-2)(3n+1)}$$

解答 $\dfrac{n}{3n+1}$

解説

$$S=\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+\cdots\cdots+\frac{1}{(3n-2)(3n+1)}$$
$$=\frac{1}{3}\left(\frac{1}{1}-\frac{1}{4}\right)+\frac{1}{3}\left(\frac{1}{4}-\frac{1}{7}\right)+\frac{1}{3}\left(\frac{1}{7}-\frac{1}{10}\right)+\cdots\cdots+\frac{1}{3}\left(\frac{1}{3n-2}-\frac{1}{3n+1}\right)$$
$$=\frac{1}{3}\left\{\left(\frac{1}{1}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{7}}\right)+\left(\cancel{\frac{1}{7}}-\cancel{\frac{1}{10}}\right)+\cdots\cdots+\left(\cancel{\frac{1}{3n-2}}-\frac{1}{3n+1}\right)\right\}$$
$$=\frac{1}{3}\left(1-\frac{1}{3n+1}\right)=\frac{1}{3}\cdot\frac{3n}{3n+1}=\frac{n}{3n+1}$$

9 和 $\dfrac{1}{2\cdot3}+\dfrac{1}{3\cdot4}+\dfrac{1}{4\cdot5}+\cdots\cdots+\dfrac{1}{(n+1)(n+2)}$ を求めよ。

解答 $\dfrac{n}{2(n+2)}$

解説

$\dfrac{1}{(k+1)(k+2)}=\dfrac{1}{k+1}-\dfrac{1}{k+2}$ であるから

$$\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\frac{1}{4\cdot5}+\cdots\cdots+\frac{1}{(n+1)(n+2)}=\left(\frac{1}{2}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}\right)$$
$$+\cdots\cdots+\left(\cancel{\frac{1}{n+1}}-\frac{1}{n+2}\right)$$
$$=\frac{1}{2}-\frac{1}{n+2}=\frac{n}{2(n+2)}$$

10 恒等式 $\dfrac{1}{(4k-3)(4k+1)}=\dfrac{1}{4}\left(\dfrac{1}{4k-3}-\dfrac{1}{4k+1}\right)$ を用いて，和

$S=\dfrac{1}{1\cdot5}+\dfrac{1}{5\cdot9}+\dfrac{1}{9\cdot13}+\cdots\cdots+\dfrac{1}{(4n-3)(4n+1)}$ を求めよ。

解答 $\dfrac{n}{4n+1}$

解説

$$S=\frac{1}{1\cdot5}+\frac{1}{5\cdot9}+\frac{1}{9\cdot13}+\cdots\cdots+\frac{1}{(4n-3)(4n+1)}$$
$$=\frac{1}{4}\left(\frac{1}{1}-\frac{1}{5}\right)+\frac{1}{4}\left(\frac{1}{5}-\frac{1}{9}\right)+\frac{1}{4}\left(\frac{1}{9}-\frac{1}{13}\right)+\cdots\cdots+\frac{1}{4}\left(\frac{1}{4n-3}-\frac{1}{4n+1}\right)$$
$$=\frac{1}{4}\left\{\left(\frac{1}{1}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{5}}-\cancel{\frac{1}{9}}\right)+\left(\cancel{\frac{1}{9}}-\cancel{\frac{1}{13}}\right)+\cdots\cdots+\left(\cancel{\frac{1}{4n-3}}-\frac{1}{4n+1}\right)\right\}$$
$$=\frac{1}{4}\left(1-\frac{1}{4n+1}\right)=\frac{1}{4}\cdot\frac{4n}{4n+1}=\frac{n}{4n+1}$$

11 数列 $\dfrac{1}{3\cdot5}$，$\dfrac{1}{5\cdot7}$，$\dfrac{1}{7\cdot9}$，……，$\dfrac{1}{(2n+1)(2n+3)}$ の和を求めよ。

解答 $\dfrac{n}{3(2n+3)}$

解説

この数列の第 k 項は $\quad\dfrac{1}{(2k+1)(2k+3)}=\dfrac{1}{2}\cdot\dfrac{(2k+3)-(2k+1)}{(2k+1)(2k+3)}$

$$=\frac{1}{2}\left(\frac{1}{2k+1}-\frac{1}{2k+3}\right)$$

求める和を S とすると

$$S=\frac{1}{2}\left\{\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{9}\right)+\cdots\cdots+\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)\right\}$$
$$=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{2n+3}\right)=\frac{n}{3(2n+3)}$$

12 数列 $\dfrac{1}{3\cdot7}$，$\dfrac{1}{7\cdot11}$，$\dfrac{1}{11\cdot15}$，…… の初項から第 n 項までの和を求めよ。

解答 $\dfrac{n}{3(4n+3)}$

解説

この数列の第 k 項は $\quad\dfrac{1}{(4k-1)(4k+3)}$

$$\frac{1}{(4k-1)(4k+3)}=\frac{1}{4}\cdot\frac{(4k+3)-(4k-1)}{(4k-1)(4k+3)}=\frac{1}{4}\left(\frac{1}{4k-1}-\frac{1}{4k+3}\right)$$

この式に $k=1,\ 2,\ \cdots\cdots,\ n$ を代入して，辺々を加えると

$$\frac{1}{3\cdot7}+\frac{1}{7\cdot11}+\frac{1}{11\cdot15}+\cdots\cdots+\frac{1}{(4n-1)(4n+3)}$$
$$=\frac{1}{4}\left(\frac{1}{3}-\frac{1}{7}\right)+\frac{1}{4}\left(\frac{1}{7}-\frac{1}{11}\right)+\frac{1}{4}\left(\frac{1}{11}-\frac{1}{15}\right)+\cdots\cdots+\frac{1}{4}\left(\frac{1}{4n-1}-\frac{1}{4n+3}\right)$$
$$=\frac{1}{4}\left\{\left(\frac{1}{3}-\cancel{\frac{1}{7}}\right)+\left(\cancel{\frac{1}{7}}-\cancel{\frac{1}{11}}\right)+\left(\cancel{\frac{1}{11}}-\cancel{\frac{1}{15}}\right)+\cdots\cdots+\left(\cancel{\frac{1}{4n-1}}-\frac{1}{4n+3}\right)\right\}$$
$$=\frac{1}{4}\left(\frac{1}{3}-\frac{1}{4n+3}\right)=\frac{1}{4}\cdot\frac{(4n+3)-3}{3(4n+3)}=\frac{n}{3(4n+3)}$$

13 次の和 S を求めよ。

$$S=\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\cdots\cdots+\frac{1}{n(n+1)}$$

解答 $\dfrac{n}{n+1}$

解説

$$S=\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\cdots\cdots+\frac{1}{n(n+1)}$$
$$=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$
$$=1-\frac{1}{n+1}=\frac{n}{n+1}$$

14 次の問いに答えよ。

(1) 恒等式 $\dfrac{1}{(2k-1)(2k+1)}=\dfrac{1}{2}\left(\dfrac{1}{2k-1}-\dfrac{1}{2k+1}\right)$ を証明せよ。

(2) 次の和 S を求めよ。

$$S=\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\cdots\cdots+\frac{1}{(2n-1)(2n+1)}$$

解答 (1) 略 (2) $\dfrac{n}{2n+1}$

解説

(1) 右辺の式を変形すると

$$\frac{1}{2}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right)=\frac{1}{2}\cdot\frac{(2k+1)-(2k-1)}{(2k-1)(2k+1)}$$
$$=\frac{1}{2}\cdot\frac{2}{(2k-1)(2k+1)}$$
$$=\frac{1}{(2k-1)(2k+1)}$$

よって $\quad\dfrac{1}{(2k-1)(2k+1)}=\dfrac{1}{2}\left(\dfrac{1}{2k-1}-\dfrac{1}{2k+1}\right)$

(2) (1) で証明した恒等式を利用すると

$$S=\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)+\frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots\cdots+\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$$
$$=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)=\frac{n}{2n+1}$$

15 次の和 S を求めよ。[各 15 点]

(1) $S=\dfrac{1}{2\cdot4}+\dfrac{1}{4\cdot6}+\dfrac{1}{6\cdot8}+\cdots\cdots+\dfrac{1}{2n\cdot2(n+1)}$

(2) $S=\dfrac{1}{\sqrt{2}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{4}}+\dfrac{1}{\sqrt{4}+\sqrt{5}}+\cdots\cdots+\dfrac{1}{\sqrt{n+1}+\sqrt{n+2}}$

解答 (1) $\dfrac{1}{2k\cdot2(k+1)}=\dfrac{1}{2}\left\{\dfrac{1}{2k}-\dfrac{1}{2(k+1)}\right\}$ であるから

$$S=\frac{1}{2}\left\{\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{6}-\frac{1}{8}\right)+\cdots\cdots+\left(\frac{1}{2n}-\frac{1}{2(n+1)}\right)\right\}$$
$$=\frac{1}{2}\left\{\frac{1}{2}-\frac{1}{2(n+1)}\right\}=\frac{n}{4(n+1)}$$

(2) $\dfrac{1}{\sqrt{k+1}+\sqrt{k+2}}=\sqrt{k+2}-\sqrt{k+1}$ であるから

$S=(\sqrt{3}-\sqrt{2})+(\sqrt{4}-\sqrt{3})+(\sqrt{5}-\sqrt{4})+\cdots\cdots+(\sqrt{n+2}-\sqrt{n+1})$
$=-\sqrt{2}+\sqrt{n+2}$

解説

(1) $\dfrac{1}{2k\cdot 2(k+1)}=\dfrac{1}{2}\left\{\dfrac{1}{2k}-\dfrac{1}{2(k+1)}\right\}$ であるから

$$S=\frac{1}{2}\left\{\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{6}-\frac{1}{8}\right)+\cdots\cdots+\left(\frac{1}{2n}-\frac{1}{2(n+1)}\right)\right\}$$
$$=\frac{1}{2}\left\{\frac{1}{2}-\frac{1}{2(n+1)}\right\}=\frac{n}{4(n+1)}$$

(2) $\dfrac{1}{\sqrt{k+1}+\sqrt{k+2}}=\sqrt{k+2}-\sqrt{k+1}$ であるから

$$S=(\sqrt{3}-\sqrt{2})+(\sqrt{4}-\sqrt{3})+(\sqrt{5}-\sqrt{4})+\cdots\cdots+(\sqrt{n+2}-\sqrt{n+1})$$
$$=-\sqrt{2}+\sqrt{n+2}$$

16 次の数列の和を求めよ。

$$\frac{1}{1\cdot 5},\ \frac{1}{5\cdot 9},\ \frac{1}{9\cdot 13},\ \cdots\cdots,\ \frac{1}{(4n-3)(4n+1)}$$

解答 $\dfrac{n}{4n+1}$

解説

この数列の第 k 項は　$\dfrac{1}{(4k-3)(4k+1)}=\dfrac{1}{4}\left(\dfrac{1}{4k-3}-\dfrac{1}{4k+1}\right)$

求める和を S とすると

$$S=\frac{1}{4}\left\{\left(\frac{1}{1}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{13}\right)+\cdots\cdots+\left(\frac{1}{4n-3}-\frac{1}{4n+1}\right)\right\}$$
$$=\frac{1}{4}\left(1-\frac{1}{4n+1}\right)=\frac{n}{4n+1}$$

17 次の和を求めよ。

(1) $\dfrac{1}{1\cdot 3}+\dfrac{1}{3\cdot 5}+\dfrac{1}{5\cdot 7}+\cdots\cdots+\dfrac{1}{49\cdot 51}$　(2) $\displaystyle\sum_{k=1}^{n}\frac{1}{(3k-1)(3k+2)}$

解答 (1) $\dfrac{25}{51}$　(2) $\dfrac{n}{2(3n+2)}$

解説

(1) 数列 $\dfrac{1}{1\cdot 3},\ \dfrac{1}{3\cdot 5},\ \dfrac{1}{5\cdot 7},\ \cdots\cdots$ の第 k 項は

$$\frac{1}{(2k-1)(2k+1)}=\frac{1}{2}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right)$$

求める和を S とすると

$$S=\frac{1}{2}\left\{\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots\cdots+\left(\frac{1}{49}-\frac{1}{51}\right)\right\}$$
$$=\frac{1}{2}\left(1-\frac{1}{51}\right)=\frac{25}{51}$$

(2) $\dfrac{1}{(3k-1)(3k+2)}=\dfrac{1}{3}\left(\dfrac{1}{3k-1}-\dfrac{1}{3k+2}\right)$

よって，求める和を S とすると

$$S=\frac{1}{3}\left\{\left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{8}\right)+\left(\frac{1}{8}-\frac{1}{11}\right)+\cdots\cdots+\left(\frac{1}{3n-1}-\frac{1}{3n+2}\right)\right\}$$
$$=\frac{1}{3}\left(\frac{1}{2}-\frac{1}{3n+2}\right)=\frac{1}{3}\cdot\frac{3n}{2(3n+2)}=\frac{n}{2(3n+2)}$$

18 次の和 S を求めよ。ただし，(2) は $n\geqq 2$ とする。

(1) $S=\dfrac{1}{1\cdot 4}+\dfrac{1}{4\cdot 7}+\dfrac{1}{7\cdot 10}+\cdots\cdots+\dfrac{1}{(3n-2)(3n+1)}$

(2) $S=\dfrac{1}{1\cdot 3}+\dfrac{1}{2\cdot 4}+\dfrac{1}{3\cdot 5}+\cdots\cdots+\dfrac{1}{n(n+2)}$

解答 (1) $S=\dfrac{n}{3n+1}$　(2) $S=\dfrac{n(3n+5)}{4(n+1)(n+2)}$

解説

(1) $\dfrac{1}{(3k-2)(3k+1)}=\dfrac{1}{3}\left(\dfrac{1}{3k-2}-\dfrac{1}{3k+1}\right)$ であるから

$$S=\frac{1}{3}\left\{\left(1-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)+\cdots\cdots+\left(\frac{1}{3n-2}-\frac{1}{3n+1}\right)\right\}$$
$$=\frac{1}{3}\left(1-\frac{1}{3n+1}\right)=\frac{n}{3n+1}$$

(2) $\dfrac{1}{k(k+2)}=\dfrac{1}{2}\left(\dfrac{1}{k}-\dfrac{1}{k+2}\right)$ であるから

$$S=\frac{1}{2}\left\{\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots\cdots+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)+\left(\frac{1}{n}-\frac{1}{n+2}\right)\right\}$$
$$=\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{1}{2}\cdot\frac{3(n+1)(n+2)-2(n+2)-2(n+1)}{2(n+1)(n+2)}$$
$$=\frac{n(3n+5)}{4(n+1)(n+2)}\quad\cdots\cdots①$$

参考 ① に $n=1$ を代入すると $\dfrac{1}{3}$ となるから，① は $n=1$ のときにも成り立つことがわかる。

19 次の数列の初項から第 n 項までの和を求めよ。

(1) $1,\ \dfrac{1}{1+2},\ \dfrac{1}{1+2+3},\ \dfrac{1}{1+2+3+4},\ \cdots\cdots$

(2) $\dfrac{3}{1^2},\ \dfrac{5}{1^2+2^2},\ \dfrac{7}{1^2+2^2+3^2},\ \dfrac{9}{1^2+2^2+3^2+4^2},\ \cdots\cdots$

(3) $\dfrac{1}{1\cdot 2\cdot 3},\ \dfrac{1}{2\cdot 3\cdot 4},\ \dfrac{1}{3\cdot 4\cdot 5},\ \dfrac{1}{4\cdot 5\cdot 6},\ \cdots\cdots$

解答 (1) $\dfrac{2n}{n+1}$　(2) $\dfrac{6n}{n+1}$　(3) $\dfrac{n(n+3)}{4(n+1)(n+2)}$

解説

(1) この数列の第 k 項は

$$\frac{1}{1+2+3+\cdots\cdots+k}=\frac{1}{\frac{1}{2}k(k+1)}=\frac{2}{k(k+1)}=2\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

よって，求める和は

$$2\left\{\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right\}$$
$$=2\left(1-\frac{1}{n+1}\right)=\frac{2n}{n+1}$$

(2) この数列の第 k 項は

$$\frac{2k+1}{\sum_{i=1}^{k}i^2}=\frac{6(2k+1)}{k(k+1)(2k+1)}=\frac{6}{k(k+1)}=6\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

よって，求める和は

$$6\left\{\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right\}$$
$$=6\left(1-\frac{1}{n+1}\right)=\frac{6n}{n+1}$$

(3) この数列の第 k 項は

$$\frac{1}{k(k+1)(k+2)}=\frac{1}{2}\left\{\frac{1}{k(k+1)}-\frac{1}{(k+1)(k+2)}\right\}$$

よって，求める和は

$$\frac{1}{2}\left[\left(\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}\right)+\left(\frac{1}{2\cdot 3}-\frac{1}{3\cdot 4}\right)+\left(\frac{1}{3\cdot 4}-\frac{1}{4\cdot 5}\right)+\cdots\cdots+\left\{\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\right\}\right]$$
$$=\frac{1}{2}\left\{\frac{1}{1\cdot 2}-\frac{1}{(n+1)(n+2)}\right\}=\frac{1}{2}\cdot\frac{(n+1)(n+2)-2}{2(n+1)(n+2)}$$
$$=\frac{n(n+3)}{4(n+1)(n+2)}$$